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Quasi-neutral limit to the drift-diffusion models for semiconductors with physical contact-insulating boundary conditions

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A B S T R A C T

In this paper the limit of vanishing Debye length in a bipolar drift-diffusion model for semiconductors with physical contact-insulating boundary conditions is studied in one-dimensional case. The quasi-neutral limit (zero-Debye-length limit) is proved by using the asymptotic expansion methods of singular perturbation theory and the classical energy methods. Our results imply that one kind of the new and interesting phenomena in semiconductor physics occurs.

1. Introduction

In this paper, we consider the semiconductor with p-n junctions insulated in one side and with contacts in the other side. The physics of p-n junction are explained by Sze [17] and Smith [16]. The scaled equations are given in the case of one space dimension as follows:

$$n_t^\lambda = \left(n_x^\lambda - n^\lambda \phi_x^\lambda\right)_x, \quad 0 < x < 1,\ t > 0, \tag{1.1}$$

$$p_t^\lambda = \left(p_x^\lambda + p^\lambda \phi_x^\lambda\right)_x, \quad 0 < x < 1,\ t > 0, \tag{1.2}$$

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$$\lambda^2\phi^\lambda_{xx} = n^\lambda - p^\lambda - D, \quad 0 < x < 1,\ t > 0 \tag{1.3}$$

with mixed Neumann–Dirichlet boundary conditions and initial conditions:

$$n^\lambda = \bar{n}(t) \geqslant 0, \qquad p^\lambda = \bar{p}(t) \geqslant 0, \qquad \phi^\lambda_x = 0, \quad x = 0,\ t > 0, \tag{1.4}$$

$$n^\lambda_x - n^\lambda\phi^\lambda_x = p^\lambda_x + p^\lambda\phi^\lambda_x = \phi^\lambda_x = 0, \quad x = 1,\ t > 0, \tag{1.5}$$

$$n^\lambda(x,0) = n^\lambda_0(x), \qquad p^\lambda(x,0) = p^\lambda_0(x), \quad 0 \leqslant x \leqslant 1. \tag{1.6}$$

The variables n^λ, p^λ, ϕ^λ are the electron density, the hole density and the electric potential, respectively. The parameter λ is the scaled Debye length of the semiconductor devices under consideration. $D = D(x)$ is the given function of space and models the doping profile (i.e., the preconcentration of electrons and holes). Because of the occurrence of p-n junctions in realistic semiconductor devices, the doping profile $D(x)$ typically changes its sign. Physically, the Dirichlet boundary conditions $n^\lambda(x=0,t) = \bar{n}(t)$, $p^\lambda(x=0,t) = \bar{p}(t)$ in (1.4) stand for the contact at the left side $x = 0$ while the boundary condition $(n^\lambda_x - n^\lambda\phi^\lambda_x)(x=1,t) = (p^\lambda_x + p^\lambda\phi^\lambda_x)(x=1,t) = \phi^\lambda_x(x=1,t) = 0$ in (1.5) stands for the insulating at the right side $x = 1$. Here $\bar{n}(t)$ and $\bar{p}(t)$ are given nonnegative functions independent of λ. For the more details of semiconductor physics, one refers to [11,10,12]. Due to the fact that if $(n^\lambda, p^\lambda, \phi^\lambda)$ is the solution to (1.1)–(1.6), then for any $c(t)$, so does $(n^\lambda, p^\lambda, \phi^\lambda + c(t))$. We re-normalized the solution to satisfy that

$$\phi^\lambda(0,t) = 0, \quad t \geqslant 0. \tag{1.7}$$

Usually semiconductor physics are concerned with large-scale structures with respect to the Debye length λ (λ takes small values, typically $\lambda^2 \approx 10^{-7}$). For such scales, the semiconductor is almost electrically neutral. This is the so-called quasi-neutrality assumption of semiconductors or plasma physics. Under the assumption of space charge neutrality, i.e., $\lambda = 0$, we formally arrive at the following quasi-neutral drift-diffusion model:

$$n^0_t = \left(n^0_x - n^0\phi^0_x\right)_x, \quad 0 < x < 1,\ t > 0, \tag{1.8}$$

$$p^0_t = \left(p^0_x + p^0\phi^0_x\right)_x, \quad 0 < x < 1,\ t > 0, \tag{1.9}$$

$$0 = n^0 - p^0 - D, \quad 0 < x < 1,\ t > 0. \tag{1.10}$$

Generally speaking, it should be expected at least formally that $(n^\lambda, p^\lambda, \phi^\lambda) \to (n^0, p^0, \phi^0)$ as $\lambda \to 0$ in the interior of the interval $[0,1]$, while it cannot be expected a priori that all of the boundary and initial conditions are maintained for the limit problem because of the singular perturbation character of the problem. The boundary conditions for (1.8)–(1.10) will be taken as

$$n^0(0,t) = \tilde{n}(t), \qquad p^0(0,t) = \tilde{p}(t), \quad t > 0, \tag{1.11}$$

$$n^0_x - n^0\phi^0_x = 0, \qquad p^0_x + p^0\phi^0_x = 0, \quad x = 1,\ t > 0, \tag{1.12}$$

where $\tilde{n}(t)$ and $\tilde{p}(t)$ will be determined later.

Similarly, we can expect a priori that (1.8)–(1.10) are supplemented by the initial data

$$n^0(x,0) = n^0_0(x), \qquad p^0(x,0) = p^0_0(x), \quad 0 \leqslant x \leqslant 1 \tag{1.13}$$

satisfying locally initial time space charge neutrality

$$n^0_0(x) - p^0_0(x) - D = 0.$$

It should point out that the quasi-neutral limit is a physically interesting (see [15]) and, but, mathematically challenging problem (see [12,20]). For the stationary drift-diffusion-Poisson models, rigorous convergence results for p-n junction devices with contacts can be found in Markowich [11] and some further extensions done by Caffarelli et al. [2] and Dolbeault et al. [3]. For the insulated conditions, there are some recent results which can be found in [5,7,6,9,8,14,19,20], et al. For the quasi-neutral limit of other models, some results can be found in [1,18], et al.

In the present paper, we concern the more deeply physical background of semiconductor and discuss the dynamic stability of semiconductor with p-n junctions and contact-insulating boundary. For the contact boundary, the structure of the boundary layer is complex and completely different from the insulated case, and, hence, it is difficult to be dealt with.

Because the physical doping profile $D(x)$ plays an important role in characterizing semiconductor, it heavily affects the dynamic stability of semiconductor devices and, thus, it yields to the complex structure of the solution to the drift-diffusion models. As the first step of studying this complex problem, we assume that $D(x)$ is a smooth C^4 function satisfying

$$D'(0)=0. \tag{1.14}$$

This avoids the occurrence of the first order boundary layer for the density. Furthermore, for simplicity, we also assume that

$$D'(1)=D'''(1)=0, \tag{1.15}$$

which avoids the occurrence of the boundary layer near the right boundary $x=1$. In fact, the structure of this kind of boundary layer in the case of the Neumann boundary condition caused by the doping profile $D(x)$ has been studied in [20].

For the boundary conditions at $x=0$, if it holds that $\bar{n}(t)-\bar{p}(t)-D(0)=0$, so-called well-prepared boundary data, there is no boundary layer near the left boundary $x=0$ because the condition $\bar{n}(t)-\bar{p}(t)-D(0)=0$ matches the inner solution (n^0,p^0,ϕ^0) satisfying the quasi-neutrality $n^0(0,t)-p^0(0,t)-D(0)=0$. In this case, we can take the inner solution as the approximating solution and the convergence from the drift-diffusion model to the quasi-neutral drift-diffusion model can be established, see Theorem 3 below.

On the other hand, if it does not hold, i.e., $\bar{n}(t)-\bar{p}(t)-D(0)\neq 0$, so-called ill-prepared boundary data, the boundary layer will appear near $x=0$. We will prove that the quasi-neutrality limit holds locally in time and that if the total density of electron and hole after doping is not very big at $x=0$, more precisely, if $|\bar{n}(t)-\bar{p}(t)-D(0)|\leqslant\eta$ for $0\leqslant\eta\leqslant\eta_0$, where $\eta_0>0$ is sufficiently small and independent of λ, then the quasi-neutrality limit holds globally in time up to the maximal existence time of the limit equations. The precise description is shown in Theorems 1 and 2. Our results imply that the structure stability of drift-diffusion models for semiconductor in the case of physical contact boundary depends upon the suitably small strength assumption on the boundary layers caused by the contact boundary. This is very different from that of insulting boundary case, where quasi-neutrality always be true no matter how strong the boundary layers caused by the insulting boundary and the doping profile is. Maybe this implies that one kind of the new and very interesting phenomena in semiconductor physics occurs.

In this paper we consider the general case, i.e., $\bar{n}(t)-\bar{p}(t)-D(0)=0$ holds or not.

For the initial value, we consider the case of no initial layer by taking the well-prepared initial data, i.e.,

$$n_0^\lambda(x)=n_0^0(x)+n_{0R},\qquad p_0^\lambda(x)=p_0^0(x)-\lambda^2\phi_{xx}^0(t=0)+p_{0R}, \tag{1.16}$$

where n_{0R}, p_{0R} will be given later. Also, the comparability condition at $(x=0,\ t=0)$

$$\bar{n}(t=0)-\bar{p}(t=0)-D(0)\big(=n_0^\lambda(0)-p_0^\lambda(0)-D(0)\big)=0 \tag{1.17}$$

is assumed. In the future paper, we will remove these technical assumptions (1.14)–(1.17) and study the dynamic stability of solutions with complex structure like mixed initial and left or right boundary layers, which could be proven by using the methods involved here combining with the careful analysis of the initial layer and boundary layer.

This paper is organized as follows: In Section 2, we state the main results of this paper. In Section 3, the approximating solutions are constructed by a method of matched asymptotic analysis. Section 4 is devoted to the proof of the main theorems.

2. Main results

In this section, we state the main results of this paper.

Assume that the initial data have an expansion as (1.16) and take the ansatz as the approximating solution

$$\begin{aligned}\left(n^\lambda_{app}, p^\lambda_{app}, \phi^\lambda_{app}\right) = \big(&n^0(x,t) + n^0_B(\xi,t) + \lambda n^1_B(\xi,t) + \lambda^2 n^2_B(\xi,t),\\ &p^0(x,t) + p^0_B(\xi,t) + \lambda p^1_B(\xi,t) + \lambda^2 p^2_B(\xi,t),\\ &\phi^0(x,t) + \phi^0_B(\xi,t)\big),\end{aligned} \tag{2.1}$$

where $\xi = \frac{x}{\lambda}$, λ is the length of the boundary layer.

First, the inner functions $(n^0, p^0, \phi^0)(x,t)$ are independent of λ and are determined as the solution of the following initial–boundary value problem:

$$n^0_t = \left(n^0_x - n^0\phi^0_x\right)_x, \quad 0 < x < 1,\ t > 0, \tag{2.2}$$

$$p^0_t = \left(p^0_x + p^0\phi^0_x\right)_x, \quad 0 < x < 1,\ t > 0, \tag{2.3}$$

$$0 = n^0 - p^0 - D, \quad 0 < x < 1,\ t > 0, \tag{2.4}$$

$$n^0(0,t) = \tilde{n}(t) = \bar{n}(t)\mathrm{e}^{\phi^0(0,t)}, \qquad p^0(0,t) = \tilde{p}(t) = \bar{p}(t)\mathrm{e}^{-\phi^0(0,t)}, \quad t > 0, \tag{2.5}$$

$$n^0_x - n^0\phi^0_x = p^0_x + p^0\phi^0_x = 0, \quad x = 1, \tag{2.6}$$

$$\left(n^0, p^0\right)(x, t=0) = \left(n^0_0(x), p^0_0(x)\right), \quad 0 < x < 1. \tag{2.7}$$

Then the boundary layer functions $(n^0_B, p^0_B, \phi^0_B)(\xi,t)$ are determined as the solution of the following problem:

$$n^0_{B,\xi} = \left(n^0(0,t) + n^0_B\right)\phi^0_{B,\xi}, \tag{2.8}$$

$$p^0_{B,\xi} = -\left(p^0(0,t) + p^0_B\right)\phi^0_{B,\xi}, \tag{2.9}$$

$$\phi^0_{B,\xi\xi} = n^0_B - p^0_B, \tag{2.10}$$

$$\phi^0_{B,\xi}(\xi = 0, t) = 0, \qquad \phi^0_B(\xi = 0, t) = -\phi^0(0,t), \tag{2.11}$$

$$\left(n^0_B, p^0_B, \phi^0_B\right) \to 0, \quad \xi \to \infty \tag{2.12}$$

and $(n^i_B, p^i_B)(\xi,t)$, $i = 1, 2$, are governed by the following problems:

$$n^1_{B,\xi} = \phi^0_x(0,t)n^0_B + \phi^0_{B,\xi}n^1_B, \tag{2.13}$$

$$p^1_{B,\xi} = -\left(\phi^0_x(0,t)p^0_B + \phi^0_{B,\xi}p^1_B\right), \tag{2.14}$$

$$\left(n_B^1, p_B^1\right) \to 0, \quad \xi \to \infty, \tag{2.15}$$

$$n_{B,t}^0 = \left[n_{B,\xi}^2 - \left(\phi_x^0(0,t)n_B^1 + \phi_{B,\xi}^0 n_B^2\right)\right]_\xi, \tag{2.16}$$

$$p_{B,t}^0 = \left[p_{B,\xi}^2 + \left(\phi_x^0(0,t)p_B^1 + \phi_{B,\xi}^0 p_B^2\right)\right]_\xi, \tag{2.17}$$

$$\left(n_B^2, p_B^2\right) \to 0, \quad \xi \to \infty. \tag{2.18}$$

The existence of the solutions for the inner equations and boundary layer equations will be discussed in Section 3.

We should point out that it follows from (2.4) and (2.5) that $\phi^0(0,t) = 0$ if $\bar{n}(t) - \bar{p}(t) - D(0) = 0$ holds, and, otherwise $\phi^0(0,t) \neq 0$. Thus, the condition $\bar{n}(t) - \bar{p}(t) - D(0) = 0$ yields that all of the boundary layer equations have only the zero solution, that is to say, there is no boundary layer. In this case, $\bar{n}(t) - \bar{p}(t) - D(0) = 0$ is called as the well-prepared boundary data. Otherwise, $\bar{n}(t) - \bar{p}(t) - D(0) \neq 0$ is called as the ill-prepared boundary data which yield to the presence of the boundary layer.

Now we introduce the new variables (z^λ, E^λ) by the transformation $z^\lambda = n^\lambda + p^\lambda$, $E^\lambda = -\phi_x^\lambda$, then (1.1)–(1.6) can be reduced to the following equivalent system for (z^λ, E^λ):

$$z_t^\lambda = \left(z_x^\lambda + DE^\lambda\right)_x - \lambda^2\left(E^\lambda E_x^\lambda\right)_x, \quad 0 \leqslant x \leqslant 1,\ t > 0, \tag{2.19}$$

$$\lambda^2\left(E_t^\lambda - E_{xx}^\lambda\right) = -\left(D_x + z^\lambda E^\lambda\right), \quad 0 \leqslant x \leqslant 1,\ t > 0, \tag{2.20}$$

$$z^\lambda = \bar{n}(t) + \bar{p}(t), \qquad E^\lambda = 0, \quad x = 0,\ t > 0, \tag{2.21}$$

$$z_x^\lambda = E^\lambda = 0, \quad x = 1,\ t > 0, \tag{2.22}$$

$$z^\lambda = z_0^\lambda(x), \qquad E^\lambda = E_0^\lambda(x), \quad t = 0, \tag{2.23}$$

where $z_0^\lambda(x) = n_0^\lambda(x) + p_0^\lambda(x)$ and $E_0^\lambda(x) = -\phi_x^\lambda(t=0)$ is given by

$$-\lambda^2 E_{0x}^\lambda(x) = n_0^\lambda(x) - p_0^\lambda(x) - D(x).$$

Here we can rewrite $E_0^\lambda(x)$ as $E_0^\lambda(x) = -\phi_x^0(t=0) + E_{0R}$, where $E_{0R}(x)$ satisfies

$$-\lambda^2 E_{0R,x} = n_{0R} - p_{0R}$$

according to (1.16). Since (1.1)–(1.6) and (2.19)–(2.23) are equivalent for the classical solutions, we can get the existence of the unique global classical solution of (2.19)–(2.23) from that of (1.1)–(1.6) (see [4]).

We note that by the transformation $z = n^0 + p^0$, $\varepsilon = -\phi_x^0$, (1.8)–(1.13) is reduced to the following equivalent system:

$$z_t = (z_x + D\varepsilon)_x, \quad 0 < x < 1,\ t > 0, \tag{2.24}$$

$$0 = -(D_x + z\varepsilon), \quad 0 < x < 1,\ t > 0, \tag{2.25}$$

$$z(0,t) = \tilde{z}(t), \quad t > 0, \tag{2.26}$$

$$z_x + D\varepsilon = 0, \quad x = 1,\ t > 0, \tag{2.27}$$

$$z(t=0) = z_0(x), \quad 0 \leqslant x \leqslant 1, \tag{2.28}$$

where $\tilde{z}(t) = \tilde{n}(t) + \tilde{p}(t) = \bar{n}(t)e^{\phi^0(0,t)} + \bar{p}(t)e^{-\phi^0(0,t)}$, $z_0(x) = n_0^0(x) + p_0^0(x)$. If $\phi^0(0,t)$ is given as a known function, then the system (2.24)–(2.28) can be solved locally or globally in time, see [20]. In fact, $\phi^0(0,t)$ can be given as

$$\phi^0(0,t) = \ln\left(\frac{D(0) + \sqrt{D^2(0) + 4\bar{n}(t)\bar{p}(t)}}{2\bar{n}(t)}\right), \quad t \geqslant 0 \tag{2.29}$$

by solving the following algebra equation:

$$\bar{n}(t)e^{\phi^0(0,t)} - \bar{p}(t)e^{-\phi^0(0,t)} - D(0) = 0, \tag{2.30}$$

which can be derived from (2.4) and (2.5). Thus, similar to the proof of Proposition 2 in [20], we have

Proposition 1 *(Existence and regularity). Let $\varepsilon(t=0) = -\frac{D'(x)}{z_0}$, $\tilde{z}(t) \geqslant \delta_0 > 0$ and $z_0 \geqslant \delta_0 > 0$ for some constant δ_0. Assume that* (1.14) *and* (1.15) *hold, and $z_0 \in C^3$ satisfies the comparability conditions*

$$z_0(x) = \tilde{z}(t=0), \quad x = 0, \tag{2.31}$$

$$z_{0x}(x) = 0, \qquad \big(z_{0x}(x) + D(x)\varepsilon(x, t=0)\big)_{xx} = 0 \quad \text{at } x = 1. \tag{2.32}$$

Then (2.24)–(2.28) *has a unique solution $(z, \varepsilon) \in C^{3,\frac{3}{2}}([0,1] \times [0,T])$, well defined in $[0,T]$ for some $T > 0$, satisfying $z \geqslant \delta > 0$ for some constant δ and $\varepsilon(x=0,t) = 0$ (according to* (1.14)*). Moreover, if $\tilde{z}(t)$ and z_0 are suitably large, then $T = +\infty$.*

Thus, by Proposition 1, one obtains the existence of the classical nonvacuum solution of (1.8)–(1.13) or (2.2)–(2.7).

Let $(z^\lambda, E^\lambda)(x,t)$ be the solution to (2.19)–(2.23) with the following initial data:

$$\big(z_0^\lambda, E_0^\lambda\big) = \left(z_0^0 + z_{0R}, \frac{-D_x(x)}{z_0} + E_{0R}\right), \tag{2.33}$$

where $z_0^0 = n_0^0(x) + p_0^0(x) - \lambda^2\phi_{xx}^0(t=0)$, $z_{0R} = n_{0R} + p_{0R}$. We impose the following decomposition for the solution:

$$\big(z^\lambda, E^\lambda\big) = \left(\big(n^0 + p^0\big) + \big(n_B^0 + p_B^0\big) + \lambda\big(n_B^1 + p_B^1\big) + \lambda^2\big(n_B^2 + p_B^2\big) + z_R, -\phi_x^0 - \frac{1}{\lambda}\phi_{B,\xi}^0 + E_R\right), \tag{2.34}$$

then we define the error term $(z_R, E_R)(x,t)$ by

$$(z_R, E_R) = \big(z^\lambda, E^\lambda\big) - \big(n_{app}^\lambda + p_{app}^\lambda, -\phi_{app,x}^\lambda\big). \tag{2.35}$$

Now we give the main results of this paper.

Theorem 1 *(The case of ill-prepared boundary data $\bar{n}(t) - \bar{p}(t) - D(0) \neq 0$ for $t > 0$: global convergence result up to the maximal existence time of the limit system). Let (z^λ, E^λ) be any solution to* (2.19)–(2.23)*. Assume that* (1.8)–(1.13) *has a sufficiently smooth solution (n, p, ϕ) with $n_0 + p_0 \geqslant \delta_0 > 0$ and well defined on $[0,1] \times [0, T_0)$ for some $0 < T_0 \leqslant \infty$. Assume also that the initial data satisfy* (2.33) *and*

$$\left\| z_{0R}(x) \right\|_{H^1} \leqslant M\lambda^2, \qquad \left\| \partial_x^2 z_{0R}(x) \right\|_{L_x^2} \leqslant M\lambda^{\frac{1}{2}}, \tag{2.36}$$

$$\left\| E_{0R}(x) \right\|_{L_x^2} \leqslant M\lambda^2, \qquad \left\| \partial_x^j E_{0R}(x) \right\|_{L_x^2} \leqslant M\lambda^{\frac{3}{2}-j}, \quad j = 1, 2. \tag{2.37}$$

Then, there exists a positive constant $\eta_0 > 0$, sufficiently small and independent of λ, such that if, for $0 \leqslant \eta \leqslant \eta_0$, it holds that

$$\left| \bar{n}(t) - \bar{p}(t) - D(0) \right| \leqslant \eta \quad \text{for all } t \in (0, T_0), \tag{2.38}$$

then, for any $T \in (0, T_0)$, there exist positive constants M and $\lambda_0, \lambda_0 \ll 1$ such that

$$\sup_{0 \leqslant t \leqslant T} \left(\left\| (z_R, E_R, z_{R,x}, z_{R,t}) \right\|_{L_x^2} + \lambda \left\| (E_R, E_{R,x}, E_{R,t}) \right\|_{L_x^2} \right) \leqslant M\sqrt{\lambda^{1-\delta}} \tag{2.39}$$

for any $\lambda \in (0, \lambda_0]$ and for any $\delta \in (0, 1)$.

Remark. Assume that $\bar{n}(t)$, $\bar{p}(t)$ are continuous functions with respect to the time t and the comparability condition for initial–boundary data $\bar{n}(0) - \bar{p}(0) - D(0) = 0$ holds, then there exist a $T^* > 0$ independent of λ and an $\eta > 0$, suitably small, such that $|\bar{n}(t) - \bar{p}(t) - D(0)| \leqslant \eta$ for all $t \in (0, T^*)$.

Thus we have

Theorem 2 *(The case of ill-prepared boundary data $\bar{n}(t) - \bar{p}(t) - D(0) \neq 0$ for $t > 0$: local convergence result). Let all assumptions of Theorem 1 hold except for (2.38). Then there exist a time T^* and constants M and $\lambda_0, \lambda_0 \ll 1$ such that, for any $T \in (0, T^*)$, (2.39) holds for any $\lambda \in (0, \lambda_0]$ and for any $\delta \in (0, 1)$.*

For the well-prepared boundary data, we have the optimal convergence rate.

Theorem 3 *(The case of well-prepared boundary data $\bar{n}(t) - \bar{p}(t) - D(0) = 0$). Let all assumptions of Theorem 1 hold except for (2.38). Then, for any $T \in (0, T_0)$, there exist positive constants M and $\lambda_0, \lambda_0 \ll 1$ such that, for any $\lambda \in (0, \lambda_0]$,*

$$\sup_{0 \leqslant t \leqslant T} \left(\left\| \left(n^\lambda - n, p^\lambda - p\right) \right\|_{L_x^2} + \lambda \left\| E^\lambda - \varepsilon \right\|_{L_x^2} \right) \leqslant M\sqrt{\lambda^{2-\delta}}, \tag{2.40}$$

for any δ with $0 < \delta < 1$.

3. Approximate solutions and matched asymptotic analysis

In this section, we construct the approximating solution including the inner functions and the boundary layer functions.

Here we set $\xi = \frac{x}{\lambda}$, where λ is the length of the boundary layer. We enforce the following decay conditions:

$$\lim_{\xi \to \infty} \left(n_B^i, p_B^i, \phi_B^0, \partial_\xi \left(n_B^i, p_B^i, \phi_B^0\right)\right)(\xi, t) = 0, \quad i = 0, 1, 2. \tag{3.1}$$

We take the following decomposition for the solution $(n^\lambda, p^\lambda, \phi^\lambda)(x, \xi, t)$ of (1.1)–(1.6):

$$\begin{aligned} \left(n^\lambda, p^\lambda, \phi^\lambda\right) = \big(& n^0(x, t) + n_B^0(\xi, t) + \lambda n_B^1(\xi, t) + \lambda^2 n_B^2(\xi, t) + n_R(x, t), \\ & p^0(x, t) + p_B^0(\xi, t) + \lambda p_B^1(\xi, t) + \lambda^2 p_B^2(\xi, t) + p_R(x, t), \\ & \phi^0(x, t) + \phi_B^0(\xi, t) + \phi_R(x, t)\big), \end{aligned} \tag{3.2}$$

then insert (3.2) into (1.1)–(1.3), by direct computations, one gets

$$\begin{aligned} & n_t^0 + n_{B,t}^0 + \lambda n_{B,t}^1 + \lambda^2 n_{B,t}^2 + n_{R,t} \\ & \quad = \left(n_x^0 - n^0 \phi_x^0\right)_x + \frac{1}{\lambda^2}\left[n_{B,\xi}^0 - \left(n^0(0,t) + n_B^0\right)\phi_{B,\xi}^0\right]_\xi \\ & \qquad + \frac{1}{\lambda}\left[n_{B,\xi}^1 - \left(\phi_x^0(0,t) n_B^0 + \phi_{B,\xi}^0 n_B^1\right)\right]_\xi + \left[n_{B,\xi}^2 - \left(\phi_x^0(0,t) n_B^1 + \phi_{B,\xi}^0 n_B^2\right)\right]_\xi \\ & \qquad + n_{R,x} + n_{B,R} + n_{RR}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} & p_t^0 + p_{B,t}^0 + \lambda p_{B,t}^1 + \lambda^2 p_{B,t}^2 + p_{R,t} \\ & \quad = \left(p_x^0 - p^0 \phi_x^0\right)_x + \frac{1}{\lambda^2}\left[p_{B,\xi}^0 + \left(p^0(0,t) + p_B^0\right)\phi_{B,\xi}^0\right]_\xi \\ & \qquad + \frac{1}{\lambda}\left[p_{B,\xi}^1 + \left(\phi_x^0(0,t) p_B^0 + \phi_{B,\xi}^0 p_B^1\right)\right]_\xi + \left[p_{B,\xi}^2 - \left(\phi_x^0(0,t) p_B^1 + \phi_{B,\xi}^0 p_B^2\right)\right]_\xi \\ & \qquad + p_{R,x} + p_{B,R} + p_{RR} \end{aligned} \tag{3.4}$$

and

$$\lambda^2\left(\phi_{xx}^0 + \frac{1}{\lambda^2}\phi_{B,\xi\xi}^0 + \phi_{R,xx}\right) = \left(n^0 - p^0 - D\right) + \left(n_B^0 - p_B^0\right) + (n_R - p_R) + \phi_{B,R}, \tag{3.5}$$

where

$$\begin{aligned} n_{B,R} &= -\frac{1}{\lambda}\left[\left(n^0(x,t) - n^0(0,t)\right)\phi_{B,\xi}^0\right]_x - \left[\left(\phi_x^0(x,t) - \phi_x^0(0,t)\right)n_B^0\right]_x \\ &\quad - \lambda\left[\left(\phi_x^0(x,t) - \phi_x^0(0,t)\right)n_B^1\right]_x - \lambda^2\left(\phi_x^0 n_B^2\right)_x, \\ n_{RR} &= \left(\left(n^0 + n_B^0 + \lambda n_B^1 + \lambda^2 n_B^2\right)\phi_{R,x} - \left(\phi_x^0 + \frac{1}{\lambda}\phi_{B,\xi}^0\right)n_R + n_R \phi_{R,x}\right)_x, \\ p_{B,R} &= -\frac{1}{\lambda}\left[\left(p^0(x,t) - p^0(0,t)\right)\phi_{B,\xi}^0\right]_x - \left[\left(\phi_x^0(x,t) - \phi_x^0(0,t)\right)p_B^0\right]_x \\ &\quad - \lambda\left[\left(\phi_x^0(x,t) - \phi_x^0(0,t)\right)p_B^1\right]_x - \lambda^2 \phi_x^0 p_B^2, \\ p_{RR} &= \left(\left(p^0 + p_B^0 + \lambda p_B^1 + \lambda^2 p_B^2\right)\phi_{R,x} - \left(\phi_x^0 + \frac{1}{\lambda}\phi_{B,\xi}^0\right)p_R + p_R \phi_{R,x}\right)_x, \\ \phi_{B,R} &= \lambda\left(n_B^1 - p_B^1\right) + \lambda^2\left(n_B^2 - p_B^2\right). \end{aligned}$$

Thus we can obtain the inner and boundary layer equations by comparing the coefficients of order $O(\lambda^k)$ of (3.3)–(3.5).

At the order λ^{-2} of (3.3), (3.4) and order λ^0 of (3.5), we get the boundary layer equations (2.8)–(2.10) for (n_B^0, p_B^0, ϕ_B^0). Using the decay conditions at infinity, one obtains that

$$n_B^0(\xi,t) = n^0(0,t)\left(\mathrm{e}^{\phi_B^0(\xi,t)} - 1\right), \qquad p_B^0(\xi,t) = p^0(0,t)\left(\mathrm{e}^{-\phi_B^0(\xi,t)} - 1\right). \tag{3.6}$$

Now we can derive the boundary conditions of the inner and boundary layer functions by letting the ansatz (2.1) of the approximating solution satisfy the boundary at $x = 0$.

The boundary condition (2.5) for $n^0(x,t)$, $p^0(x,t)$ at $x=0$ can be derived as follows. By setting the zero order approximation of $(n^\lambda_{app}, p^\lambda_{app}, \phi^\lambda_{app})|_{x=0}$ to be $(\bar n(t), \bar p(t), 0)$, one gets the following relations:

$$n^0(0,t) + n^0_B(0,t) = \bar n(t), \qquad p^0(0,t) + p^0_B(0,t) = \bar p(t), \quad t>0, \tag{3.7}$$

$$\phi^0(0,t) + \phi^0_B(0,t) = 0, \quad t>0. \tag{3.8}$$

Then it follows from (3.6) and (3.8) that

$$n^0_B(0,t) = n^0(0,t)\big(e^{-\phi^0(0,t)} - 1\big), \tag{3.9}$$

$$p^0_B(0,t) = p^0(0,t)\big(e^{\phi^0(0,t)} - 1\big). \tag{3.10}$$

Solving the algebra equations (3.7), (3.9) and (3.10), one gives (2.5).

We impose on $\phi^\lambda_{app,x}(0,t) = 0$, then we can get

$$\phi^0_x(0,t) = 0, \qquad \phi^0_{B,\xi}(0,t) = 0, \quad t>0. \tag{3.11}$$

The boundary condition $\phi^0_x(0,t) = 0$ can be guaranteed by (1.14). Otherwise, a boundary layer of first order is required to correct $\phi^0_x(0,t) \neq 0$.

Thus, we collect the boundary layer equations for $\phi^0_B(\xi,t)$ by the following system:

$$\phi^0_{B,\xi\xi} = n^0(0,t)\big(e^{\phi^0_B} - 1\big) - p^0(0,t)\big(e^{-\phi^0_B} - 1\big), \tag{3.12}$$

$$\phi^0_{B,\xi}(0,t) = 0, \qquad \phi^0_B(0,t) = -\phi^0(x=0,t), \tag{3.13}$$

$$\phi^0_B \to 0, \quad \xi \to \infty. \tag{3.14}$$

For the case of well-prepared boundary data, i.e., $\bar n(t) - \bar p(t) - D(0) = 0$, it follows from (2.29) that $\phi^0(0,t) = 0$, which yields that all boundary layer functions are zero functions by solving the boundary layer equations. In particular, the comparability condition (1.17) implies that $\phi^0(0,0) = 0$, and hence

$$\phi^0_B(\xi,0) = 0 \tag{3.15}$$

by solving (3.12)–(3.14). And we can take

$$\big(n^i_B, p^i_B\big)(\xi,0) = 0, \quad i = 0,1,2, \tag{3.16}$$

by (2.8)–(2.18) and (3.15).

Moreover, if $|\bar n(t) - \bar p(t) - D(0)| = \eta_1$, then it follows from (2.29) that there exists a constant $C(T_0) > 0$, depending only upon the bounds of $\bar n(t)$, $\bar p(t)$ and $D(0)$, such that it holds that, for any $\eta_1 > 0$ sufficiently small,

$$\big|\phi(0,t)\big| \leqslant C\eta_1, \quad 0 \leqslant t \leqslant T_0,$$

so when η in (2.38) is taken small enough, $|\phi(0,t)|$ is sufficiently small and controlled by the upper bound η only if $\eta_1 \leqslant \eta$. Meanwhile, by (3.13) and the monotonicity of ϕ^0_B [13], we have that ϕ^0_B, $\phi^0_{B,\xi}$, $\phi^0_{B,\xi\xi}$ are also sufficiently small; and from (3.6), we can obtain that n^0_B, p^0_B are sufficiently small.

At the order λ^{-1} and the order λ^0 of (3.3), (3.4), we get (2.13), (2.14) for (n^1_B, p^1_B) and (2.16), (2.17) for (n^2_B, p^2_B), respectively. Note that they are all ODE, together with the decay conditions at infinity, the equations for (n_B^i, p_B^i), $i = 1, 2$, are easy to be solved only if one solves (3.12)–(3.14). The existence and exponential decay rate at $\xi \to \infty$ for (3.12)–(3.14) had been established by Markowich et al. in [11,10].

Now we state some properties of the boundary layers.

Lemma 1. *Assume that the inner solution* (n^0, p^0) *is* $C^\infty([0,1] \times [0,T])$ *for any* $T > 0$*, then there exist positive constants* M *and* η*, independent of* λ*, such that*

$$\left\| \partial_t^{k_1} \left(\xi^{k_2} \partial_\xi^{k_3} \left(n_B^i, p_B^i, \phi_B^0 \right) \right) \right\|_{L^\infty_{(x,t)}([0,1]\times[0,T])} \leqslant M\eta, \quad i = 0, 1, 2, \tag{3.17}$$

$$\left\| \partial_t^{k_1} \left(\xi^{k_2} \partial_\xi^{k_3} \left(n_B^i, p_B^i, \phi_B^0 \right) \right) \right\|_{L_t^\infty([0,T]; L_x^2([0,1]))} \leqslant M\lambda^{\frac{1}{2}}, \quad i = 0, 1, 2, \tag{3.18}$$

for any nonnegative integer k_j*,* $j = 1, 2, 3$*. Moreover, under the assumption of smallness on* η *in Theorem* 1*,* η *in* (3.17) *can be sufficiently small.*

4. Energy estimates

In this section, we prove the main results by introducing two Liapunov-type functionals and the careful energy method.

Assume that the initial datum has the form of (2.33), replacing (z^λ, E^λ) by

$$\begin{aligned} \left(z^\lambda, E^\lambda\right) = \Bigg(& \left(n^0 + p^0\right) + h(x)\left(n_B^0 + p_B^0\right) + \lambda h(x)\left(n_B^1 + p_B^1\right) + \lambda^2 h(x)\left(n_B^2 + p_B^2\right) + z_R, \\ & -\phi_x^0 - \frac{1}{\lambda} h(x)\phi_{B,\xi}^0 + E_R \Bigg), \end{aligned} \tag{4.1}$$

where $h(x)$ is a smooth C^2 cut-off function satisfying $h(0) = 1$ and $h(1) = h'(1) = h''(1) = h'(0) = h''(0) = 0$. From (2.23), (2.28), (2.33)–(2.35), (3.15), (3.16), one can get the initial values of (z_R, E_R) as

$$\begin{aligned} z_R(t=0) &= z^\lambda(t=0) - \left(n_{app}^\lambda + p_{app}^\lambda\right)(t=0) \\ &= z_0^\lambda - \left(\left(n^0 + p^0\right) + \left(n_B^0 + p_B^0\right) + \lambda\left(n_B^1 + p_B^1\right) + \lambda^2\left(n_B^2 + p_B^2\right)\right)(t=0) \\ &= z_{0R} - \lambda^2 \phi_{xx}^0(t=0), \end{aligned} \tag{4.2}$$

$$\begin{aligned} E_R(t=0) &= E^\lambda(t=0) + \phi_{app,x}^\lambda(t=0) \\ &= \frac{-D_x(x)}{z_0} + E_{0R} + \phi_x^0(t=0) + \frac{1}{\lambda}\phi_{B,\xi}^0(t=0) \\ &= E_{0R}. \end{aligned} \tag{4.3}$$

Inserting (4.1) into the system (2.19)–(2.20) and using the equations of the inner and boundary layers, we have

$$z_{R,t} = A_{1,x} + A_{2,x} + f, \quad 0 < x < 1,\ t > 0, \tag{4.4}$$

$$\begin{aligned} & \lambda^2 (E_{R,t} - E_{R,xx}) + \left(\left(n^0 + p^0\right) + h\left(n_B^0 + p_B^0\right) + \lambda h\left(n_B^1 + p_B^1\right) + \lambda^2 h\left(n_B^2 + p_B^2\right)\right) E_R \\ & \quad = g_1 + g_2, \quad 0 < x < 1,\ t > 0, \end{aligned} \tag{4.5}$$

where

$$
\begin{aligned}
A_1 &= z_{R,x} + DE_R + \lambda^2\left(\phi_x^0 + \frac{h}{\lambda}\phi_{B,\xi}^0\right)E_{R,x} + \lambda^2\left(\phi_{xx}^0 + \frac{h}{\lambda^2}\phi_{B,\xi\xi}^0\right)E_R - \lambda^2 E_R E_{R,x},\\
A_2 &= -\frac{h}{\lambda}\big(\big(n^0 - n^0(0,t)\big) - \big(p^0 - p^0(0,t)\big)\big)\phi_{B,\xi}^0 + \frac{h-h^2}{\lambda}\phi_{B,\xi}^0\big(n_B^0 - p_B^0\big)\\
&\quad - h\big(\phi_x^0 - \phi_x^0(0,t)\big)\big(n_B^0 - p_B^0\big) + h\phi_{B,\xi}^0\big(n_B^1 - p_B^1\big)\\
&\quad + \lambda h\phi_x^0(0,t)\big(n_B^1 - p_B^1\big) + \lambda h\phi_{B,\xi}^0\big(n_B^2 - p_B^2\big) - \lambda^2\phi_x^0\phi_{xx}^0 - \lambda h\phi_{xx}^0\phi_{B,\xi}^0\\
&\quad + h'\sum_{i=0}^{2}\lambda^i\big(n_B^i + p_B^i\big) + \frac{h'}{\lambda}\phi_{B,\xi}^0\left(\phi_x^0 + \frac{h}{\lambda}\phi_{B,\xi}^0\right),\\
f &= -\big(\lambda h\big(n_{B,t}^1 + p_{B,t}^1\big) + \lambda^2 h\big(n_{B,t}^2 + p_{B,t}^2\big)\big),\\
g_1 &= \left(\phi_x^0 + \frac{h}{\lambda}\phi_{B,\xi}^0\right)z_R - z_R E_R,\\
g_2 &= \frac{h}{\lambda}\phi_{B,\xi}^0\big(\big(n^0 - n^0(0,t)\big) + \big(p^0 - p^0(0,t)\big)\big) + h\big(\phi_x^0 - \phi_x^0(0,t)\big)\big(n_B^0 + p_B^0\big)\\
&\quad + \frac{h^2-h}{\lambda}\phi_{B,\xi}^0\big(n_B^0 + p_B^0\big) + h\big(n_{B,\xi}^1 - p_{B,\xi}^1\big) + \lambda h\phi_x^0\big(n_B^1 + p_B^1\big)\\
&\quad + \lambda^2 h\phi_x^0\big(n_B^2 + p_B^2\big) + \lambda h^2\phi_{B,\xi}^0\big(n_B^2 + p_B^2\big) + \lambda^2\phi_{xt}^0 + \lambda h\phi_{B,\xi t}^0\\
&\quad - \lambda^2\phi_{xxx}^0 + \big(h^2 - h\big)\phi_{B,\xi}^0\big(n_B^1 - p_B^1\big) - 2h'\phi_{B,\xi\xi}^0 - \lambda h''\phi_{B,\xi}^0.
\end{aligned}
$$

To perform the energy estimates, we derive some boundary conditions for the error functions. From (2.21), (2.34), (3.11) and (3.7), one gets

$$(z_R, E_R)(x=0,t) = 0. \tag{4.6}$$

Since $D_x(x=1) = 0$, one gets from (2.4), (2.6) that

$$\phi_x^0(x=1,t) = 0 \tag{4.7}$$

and hence from (2.22), (2.34),

$$E_R(x=1,t) = 0. \tag{4.8}$$

Since $(A_1 + A_2)(x,t)$ can be rewritten as

$$
\begin{aligned}
A_1 + A_2 &= z_{R,x} + DE_R + \frac{h}{\lambda}\big(n_{B,\xi}^0 + p_{B,\xi}^0\big) + h\big(n_{B,\xi}^1 + p_{B,\xi}^1\big) - \frac{h}{\lambda}D(x)\phi_{B,\xi}^0\\
&\quad - \lambda^2 E^\lambda E_x^\lambda + \lambda h\phi_x^0(0,t)\big(n_B^1 - p_B^1\big) + \lambda h\phi_{B,\xi}^0\big(n_B^2 - p_B^2\big),
\end{aligned}
$$

from (4.1), (2.22), (1.12), (1.10), (1.15), (2.25), one gets

$$(A_1 + A_2)(x=1,t) = 0. \tag{4.9}$$

We begin our proof by borrowing the following estimate of Growall's inequality's type from the paper [20].

Lemma 2. *Let $\Gamma^\lambda(t)$, $G^\lambda(t)$ be nonnegative functions satisfying*

$$\Gamma^\lambda(t)+\int_0^t G^\lambda(s)\,ds \leqslant M\Gamma^\lambda(t=0)+M\int_0^t \left(\Gamma^\lambda(s)+\left(\Gamma^\lambda(s)\right)^2\right)ds + M\int_0^t \Gamma^\lambda(s)G^\lambda(s)\,ds + M\left(\Gamma^\lambda(t)\right)^2 + M\lambda,$$

where M are some positive constants independent of λ. Then for any $T\in[0,T_{max})$, $T_{max}\leqslant\infty$, there exists a $\lambda_0\ll 1$ such that, for any λ: $0<\lambda\leqslant\lambda_0$, if $\Gamma^\lambda(t=0)\leqslant\tilde{M}\lambda^{\min\{\alpha,1\}}$ for some $\alpha>0$, then

$$\Gamma^\lambda(t)\leqslant\tilde{M}\lambda^{\min\{\alpha,1\}-\delta}$$

holds for some constant $\tilde{M}$ independent of λ and any $\delta\in(0,\min\{\alpha,1\})$ and $0\leqslant t\leqslant T$.

In the following, we use c_i, δ_i, ϵ, and $M(\epsilon)$ or M to denote the constants which are independent of λ and may differ from one line to another.

Now we start the energy estimates.

Lemma 3. *Under the assumptions of Theorem 1, we have*

$$\begin{aligned}
&\left\|z_R(t)\right\|_{L_x^2}^2+\lambda^2\left\|E_R(t)\right\|_{L_x^2}^2+\int_0^t\left\|(z_{R,x},E_R)\right\|_{L_x^2}^2dt+\lambda^2\int_0^t\|E_{R,x}\|_{L_x^2}^2dt\\
&\quad\leqslant\left\|z_R(x,0)\right\|_{L_x^2}^2+\lambda^2\left\|E_R(x,0)\right\|_{L_x^2}^2\\
&\qquad+M\int_0^t\|z_R\|_{L_x^2}^2dt+M\lambda^4\int_0^t\left\|(E_R,E_{R,x})\right\|_{L_x^2}^2\|E_{R,x}\|_{L_x^2}^2dt\\
&\qquad+M\int_0^t\left\|(z_R,z_{R,x})\right\|_{L_x^2}^2\|E_R\|_{L_x^2}^2dt+M\lambda.
\end{aligned}\tag{4.10}$$

Proof. Multiplying (4.4) by z_R and integrating the resulting equation over $[0,1]$ with respect to x, by (4.6), (4.9) and integrations by parts, one gets

$$\frac{1}{2}\frac{d}{dt}\|z_R\|_{L_x^2}^2=-\int_0^1 A_1z_{R,x}\,dx-\int_0^1 A_2z_{R,x}\,dx+\int_0^1 fz_R\,dx,$$

that is,

$$\frac{1}{2}\frac{d}{dt}\|z_R\|_{L_x^2}^2 + \|z_{R,x}\|_{L_x^2}^2$$
$$= -\int_0^1 DE_R z_{R,x}\,dx + \int_0^1 f z_R\,dx - \int_0^1 A_2 z_{R,x}\,dx$$
$$- \int_0^1 \lambda^2\left[\left(\phi_x^0 + \frac{h}{\lambda}\phi_{B,\xi}^0\right)E_{R,x} + \left(\phi_{xx}^0 + \frac{h}{\lambda^2}\phi_{B,\xi\xi}^0\right)E_R - E_R E_{R,x}\right] z_{R,x}\,dx. \tag{4.11}$$

Now we estimate each term in the right-hand side of (4.11).

First, by the Cauchy–Schwarz inequality and the definition of f, one obtains

$$-\int_0^1 DE_R z_{R,x}\,dx \leqslant \epsilon\|z_{R,x}\|_{L_x^2}^2 + M(\epsilon)\|E_R\|_{L_x^2}^2, \tag{4.12}$$

$$\int_0^1 f z_R\,dx \leqslant \epsilon\|z_R\|_{L_x^2}^2 + M\lambda^3. \tag{4.13}$$

Here and in the following ϵ is any constant independent of λ.

For the third term (called I_3) in the right side, we have

$$I_3 \leqslant \epsilon\|z_{R,x}\|_{L_x^2}^2 + M\lambda, \tag{4.14}$$

where we have used the properties of the boundary layer functions and

$$\int_0^1 \left|\frac{h}{\lambda}\big(n^0 - n^0(0,t)\big)\phi_{B,\xi}^0\right|^2 dx = \int_0^1 \left|\int_0^1 h\partial_x n^0(\theta x)\,d\theta\,\frac{x}{\lambda}\phi_{B,\xi}^0\right|^2 dx$$
$$\leqslant M\lambda,$$

$$\int_0^1 \left|\frac{h-h^2}{\lambda}\phi_{B,\xi}^0\big(n_B^0 - p_B^0\big)\right|^2 dx = \int_0^1 \left|\frac{(h-h^2)-(h(0)-h^2(0))}{\lambda}\phi_{B,\xi}^0\big(n_B^0 - p_B^0\big)\right|^2 dx$$
$$= \int_0^1 \left|\int_0^1 \partial_x\big(h-h^2\big)(\theta x)\,d\theta\,\frac{x}{\lambda}\phi_{B,\xi}^0\big(n_B^0 - p_B^0\big)\right|^2 dx$$
$$\leqslant M\lambda,$$

$$\int_0^1 \left|\frac{h'}{\lambda}\phi_{B,\xi}^0\left(\phi_x^0 + \frac{h}{\lambda}\phi_{B,\xi}^0\right)\right|^2 dx = \int_0^1 \left|\frac{h'}{x}\frac{x}{\lambda}\phi_{B,\xi}^0\phi_x^0 + \frac{hh'}{x^2}\left(\frac{x}{\lambda}\phi_{B,\xi}^0\right)^2\right|^2 dx$$
$$\leqslant M\lambda$$

by the Hardy–Littlewood inequality and $h'(0) = h''(0) = 0$.

For the forth term (called I_4), by Cauchy–Schwarz inequality and Sobolev's Lemma, we have

$$\begin{aligned} I_4 \leqslant{} & \epsilon \|z_{R,x}\|_{L_x^2}^2 + M\lambda^4 \|E_R\|_{L_x^2}^2 + M\lambda^4 \|E_{R,x}\|_{L_x^2}^2 + M \big\|\phi^0_{B,\xi\xi}\big\|^2_{L^\infty_{xt}} \|E_R\|^2_{L^2_x} \\ & + M\lambda^2 \big\|\phi^0_{B,\xi}\big\|^2_{L^\infty_{xt}} \|E_{R,x}\|^2_{L^2_x} + M\lambda^4 \big\|(E_R, E_{R,x})\big\|^2_{L^2_x} \|E_{R,x}\|^2_{L^2_x}. \end{aligned} \tag{4.15}$$

Thus, combining (4.11)–(4.15) and taking ϵ, δ small enough, one gets

$$\begin{aligned} \frac{d}{dt}\|z_R\|^2_{L^2_x} + c_1 \|z_{R,x}\|^2_{L^2_x} \leqslant{} & M\|E_R\|^2_{L^2_x} + \epsilon \|z_R\|^2_{L^2_x} + M\lambda^4 \big\|(E_R, E_{R,x})\big\|^2_{L^2_x} \\ & + M \big\|\phi^0_{B,\xi\xi}\big\|^2_{L^\infty_{xt}} \|E_R\|^2_{L^2_x} + M\lambda^2 \big\|\phi^0_{B,\xi}\big\|^2_{L^\infty_{xt}} \|E_{R,x}\|^2_{L^2_x} \\ & + M\lambda^4 \big\|(E_R, E_{R,x})\big\|^2_{L^2_x} \|E_{R,x}\|^2_{L^2_x} + M\lambda. \end{aligned} \tag{4.16}$$

Integrating (4.16) with respect to t over $[0, t]$, one gets

$$\begin{aligned} & \big\|z_R(t)\big\|^2_{L^2_x} + c_1 \int_0^t \|z_{R,x}\|^2_{L^2_x}\,dt \\ & \leqslant \big\|z_R(x,0)\big\|^2_{L^2_x} + M\int_0^t \|E_R\|^2_{L^2_x}\,dt + \epsilon \int_0^t \|z_R\|^2_{L^2_x}\,dt + M\lambda^4 \int_0^t \|E_{R,x}\|^2_{L^2_x}\,dt \\ & \quad + M \big\|\phi^0_{B,\xi\xi}\big\|^2_{L^\infty_{xt}} \int_0^t \|E_R\|^2_{L^2_x}\,dt + M\lambda^2 \big\|\phi^0_{B,\xi}\big\|^2_{L^\infty_{xt}} \int_0^t \|E_{R,x}\|^2_{L^2_x}\,dt \\ & \quad + M\lambda^4 \int_0^t \big\|(E_R, E_{R,x})\big\|^2_{L^2_x} \|E_{R,x}\|^2_{L^2_x}\,dt + M\lambda. \end{aligned} \tag{4.17}$$

Multiplying (4.5) by E_R and integrating the resulting equation over $[0, 1]$ with respect to x, by (4.6), (4.8) and integrations by parts, one gets

$$\begin{aligned} & \frac{\lambda^2}{2}\frac{d}{dt}\|E_R\|^2_{L^2_x} + \lambda^2 \|E_{R,x}\|^2_{L^2_x} \\ & \quad + \int_0^1 \Big(\big(n^0 + p^0\big) + h\big(n^0_B + p^0_B\big) + \lambda h\big(n^1_B + p^1_B\big) + \lambda^2 h\big(n^2_B + p^2_B\big)\Big)|E_R|^2\,dx \\ & = \int_0^1 (g_1 + g_2) E_R\,dx. \end{aligned} \tag{4.18}$$

By the Cauchy–Schwarz inequality, we get

$$\int_0^1 (g_1+g_2)E_R\,dx \leqslant \epsilon\|E_R\|_{L_x^2}^2 + M\|z_R\|_{L_x^2}^2 + M\left\|\xi\phi_{B,\xi}^0\right\|_{L_{xt}^\infty}^2\|z_{R,x}\|_{L_x^2}^2$$
$$+ M\left\|(z_R,z_{R,x})\right\|_{L_x^2}^2\|E_R\|_{L_x^2}^2 + M\lambda, \tag{4.19}$$

where we have used Hardy–Littlewood's inequality and

$$\begin{aligned}\int_0^1 \frac{h}{\lambda}\phi_{B,\xi}^0 z_R E_R\,dx &= \int_0^1 h\frac{x}{\lambda}\phi_{B,\xi}^0 E_R\frac{z_R}{x}\,dx\\ &\leqslant \left\|h\xi\phi_{B,\xi}^0\right\|_{L_x^\infty}\|E_R\|_{L_x^2}\left\|\frac{z_R}{x}\right\|_{L_x^2}\\ &= \left\|h\xi\phi_{B,\xi}^0\right\|_{L_x^\infty}\|E_R\|_{L_x^2}\left\|\frac{z_R-z_R(x=0)}{x}\right\|_{L_x^2}\\ &\leqslant \epsilon\|E_R\|_{L_x^2}^2 + M\left\|\xi\phi_{B,\xi}^0\right\|_{L_{xt}^\infty}^2\|z_{R,x}\|_{L_x^2}^2\end{aligned}$$

thanks to (4.6).

Then from (4.18), (4.19) and taking ϵ small enough, with the positivity of n^0, p^0, one gets

$$\lambda^2\frac{d}{dt}\|E_R\|_{L_x^2}^2 + \lambda^2\|E_{R,x}\|_{L_x^2}^2 + c_2\|E_R\|_{L_x^2}^2 \leqslant M\|z_R\|_{L_x^2}^2 + M\left\|\xi\phi_{B,\xi}^0\right\|_{L_{xt}^\infty}^2\|z_{R,x}\|_{L_x^2}^2$$
$$+ M\left\|(z_R,z_{R,x})\right\|_{L_x^2}^2\|E_R\|_{L_x^2}^2 + M\lambda. \tag{4.20}$$

Integrating (4.20) with respect to t, one gets

$$\lambda^2\left\|E_R(t)\right\|_{L_x^2}^2 + \lambda^2\int_0^t \|E_{R,x}\|_{L_x^2}^2\,dt + c_2\int_0^t \|E_R\|_{L_x^2}^2\,dt$$
$$\leqslant \lambda^2\left\|E_R(x,0)\right\|_{L_x^2}^2 + M\int_0^t \|z_R\|_{L_x^2}^2\,dt + M\left\|\xi\phi_{B,\xi}^0\right\|_{L_{xt}^\infty}^2\int_0^t \|z_{R,x}\|_{L_x^2}^2\,dt$$
$$+ M\int_0^t \left\|(z_R,z_{R,x})\right\|_{L_x^2}^2\|E_R\|_{L_x^2}^2\,dt + M\lambda. \tag{4.21}$$

Then $\delta(4.17)+(4.21)$, by taking $\delta>0$ small enough and $M\|\xi\phi_{B,\xi}^0\|_{L_{xt}^\infty}^2 \leqslant \frac{\delta c_1}{2}$, $M\|\phi_{B,\xi}^0\|_{L_{xt}^\infty}^2 \leqslant \frac{1}{2}$, which can be guaranteed by (3.17) by choosing η to be sufficiently small, we can get that $M\|\xi\phi_{B,\xi}^0\|_{L_{xt}^\infty}^2\int_0^t \|z_{R,x}\|_{L_x^2}^2\,dt$ and $M\lambda^2\|\phi_{B,\xi}^0\|_{L_{xt}^\infty}^2\int_0^t \|E_{R,x}\|_{L_x^2}^2\,dt$ in the right-hand side can be absorbed by $\delta c_1\int_0^t \|z_{R,x}\|_{L_x^2}^2\,dt$ and $\lambda^2\int_0^t \|E_{R,x}\|_{L_x^2}^2\,dt$ in the left-hand side. Then we can obtain (4.10). The proof of Lemma 3 is completed. $\square$

Next we show the estimates of the time derivatives $(z_{R,t}, E_{R,t})$.

Lemma 4. *Under the assumptions of Theorem* 1, *we have*

$$\begin{aligned}
&\left\|z_{R,t}(t)\right\|_{L_x^2}^2+\lambda^2\left\|E_{R,t}(t)\right\|_{L_x^2}^2+\lambda^2\int_0^t\|E_{R,xt}\|_{L_x^2}^2\,dt+\int_0^t\left\|(z_{R,xt},E_{R,t})\right\|_{L_x^2}^2\,dt\\
&\quad\leqslant M\left(\left\|z_{R,t}(x,0)\right\|_{L_x^2}^2+\lambda^2\left\|E_{R,t}(x,0)\right\|_{L_x^2}^2\right)+M\lambda^2\int_0^t\|E_{R,x}\|_{L_x^2}^2\,dt\\
&\qquad+M\int_0^t\left\|(E_R,z_R,z_{R,t},z_{R,x})\right\|_{L_x^2}^2\,dt+M\lambda^4\int_0^t\left\|(E_{R,t},E_{R,xt})\right\|_{L_x^2}^2\,dt\\
&\qquad+M\lambda^4\int_0^t\left(\left\|(E_{R,t},E_{R,xt})\right\|_{L_x^2}^2\|E_{R,x}\|_{L_x^2}^2+\|E_R\|_{L_x^2}^2\|E_{R,xt}\|_{L_x^2}^2\right)dt\\
&\qquad+M\int_0^t\left(\left\|(z_{R,t},z_{R,xt})\right\|_{L_x^2}^2\|E_R\|_{L_x^2}^2+\left\|(z_R,z_{R,x})\right\|_{L_x^2}^2\|E_{R,t}\|_{L_x^2}^2\right)dt\\
&\qquad+M\lambda.
\end{aligned}\tag{4.22}$$

Proof. Differentiating (4.4) with respect to t, multiplying the resulting equations by $z_{R,t}$, then integrating it over $[0,1]\times[0,t]$ and noting that

$$z_{R,t}(x=0,t)=0,\qquad (A_1+A_2)_t(x=1,t)=0,$$

one gets by integration by parts that

$$\begin{aligned}
&\frac{1}{2}\left\|z_{R,t}(t)\right\|_{L_x^2}^2+\int_0^t\|z_{R,xt}\|_{L_x^2}^2\,dt\\
&\quad=\frac{1}{2}\left\|z_{R,t}(t=0)\right\|_{L_x^2}^2+\int_0^t\int_0^1 f_t z_{R,t}\,dx\,dt-\int_0^t\int_0^1 DE_{R,t}z_{R,xt}\,dx\,dt\\
&\qquad-\int_0^t\int_0^1\lambda^2\left[\left(\phi_x^0+\frac{h}{\lambda}\phi_{B,\xi}^0\right)E_{Rx}+\left(\phi_{xx}^0+\frac{h}{\lambda^2}\phi_{B,\xi\xi}^0\right)E_R-E_RE_{R,x}\right]_t z_{R,xt}\,dx\,dt\\
&\qquad-\int_0^t\int_0^1 A_{2,t}z_{R,xt}\,dx\,dt.
\end{aligned}\tag{4.23}$$

First, by the Cauchy–Schwarz inequality, one gets

$$\int_0^t\int_0^1 f_t z_{R,t}\,dx\,dt\leqslant\epsilon\int_0^t\|z_{R,t}\|_{L_x^2}^2\,dt+M\lambda^2,\tag{4.24}$$

$$-\int_0^t\int_0^1 DE_{R,t}z_{R,xt}\,dx\,dt \leqslant \epsilon\int_0^t \|z_{R,xt}\|_{L_x^2}^2\,dt + M\int_0^t \|E_{R,t}\|_{L_x^2}^2\,dt. \tag{4.25}$$

For the forth term (called I_4) in the right hand, one obtains that

$$\begin{aligned}
I_4 \leqslant{}& \epsilon\int_0^t \|z_{R,xt}\|_{L_x^2}^2\,dt + M\lambda^4\int_0^t \big\|(E_R, E_{R,x}, E_{R,t}, E_{R,xt})\big\|_{L_x^2}^2\,dt \\
&+ M\big\|\phi^0_{B,\xi\xi t}\big\|_{L^\infty_{xt}}^2\int_0^t \|E_R\|_{L_x^2}^2\,dt + M\big\|\phi^0_{B,\xi\xi}\big\|_{L^\infty_{xt}}^2\int_0^t \|E_{R,t}\|_{L_x^2}^2\,dt \\
&+ M\lambda^2\big\|\phi^0_{B,\xi t}\big\|_{L^\infty_{xt}}^2\int_0^t \|E_{R,x}\|_{L_x^2}^2\,dt + M\lambda^2\big\|\phi^0_{B,\xi}\big\|_{L^\infty_{xt}}^2\int_0^t \|E_{R,xt}\|_{L_x^2}^2\,dt \\
&+ M\lambda^4\int_0^t \Big(\big\|(E_{R,t}, E_{R,xt})\big\|_{L_x^2}^2\|E_{R,x}\|_{L_x^2}^2 + \|E_R\|_{L_x^2}^2\|E_{R,xt}\|_{L_x^2}^2\Big)\,dt.
\end{aligned} \tag{4.26}$$

For the fifth term (called I_5), one obtains that

$$I_5 \leqslant \epsilon\int_0^t \|z_{R,xt}\|_{L_x^2}^2\,dt + M\lambda, \tag{4.27}$$

where we have used the properties of the boundary layer functions and Hardy–Littlewood inequality.

Therefore, combining (4.23)–(4.27) and taking ϵ small enough, one shows that

$$\begin{aligned}
\big\|z_{R,t}(t)\big\|_{L_x^2}^2 + c_3\int_0^t \|z_{R,xt}\|_{L_x^2}^2\,dt \leqslant{}& \big\|z_{R,t}(x,0)\big\|_{L_x^2}^2 + M\int_0^t \|E_{R,t}\|_{L_x^2}^2\,dt + \epsilon\int_0^t \|z_{R,t}\|_{L_x^2}^2\,dt \\
&+ M\lambda^4\int_0^t \big\|(E_R, E_{R,x}, E_{R,t}, E_{R,xt})\big\|_{L_x^2}^2\,dt \\
&+ M\big\|\phi^0_{B,\xi\xi t}\big\|_{L^\infty_{xt}}^2\int_0^t \|E_R\|_{L_x^2}^2\,dt + M\big\|\phi^0_{B,\xi\xi}\big\|_{L^\infty_{xt}}^2\int_0^t \|E_{R,t}\|_{L_x^2}^2\,dt \\
&+ M\lambda^2\big\|\phi^0_{B,\xi t}\big\|_{L^\infty_{xt}}^2\int_0^t \|E_{R,x}\|_{L_x^2}^2\,dt + M\lambda^2\big\|\phi^0_{B,\xi}\big\|_{L^\infty_{xt}}^2\int_0^t \|E_{R,xt}\|_{L_x^2}^2\,dt \\
&+ M\lambda^4\int_0^t \Big(\big\|(E_{R,t}, E_{R,xt})\big\|_{L_x^2}^2\|E_{R,x}\|_{L_x^2}^2 + \|E_R\|_{L_x^2}^2\|E_{R,xt}\|_{L_x^2}^2\Big)\,dt \\
&+ M\lambda.
\end{aligned} \tag{4.28}$$

Note that $E_{R,t}$ satisfies

$$E_{R,t}(x=0,1;t)=0, \quad t>0,$$

thus, differentiating (4.5) with respect to t, multiplying the resulting equations by $E_{R,t}$, and then integrating it over $[0,1]\times[0,t]$, one gets by integrations by parts that

$$\begin{aligned}
&\frac{\lambda^2}{2}\left\|E_{R,t}(t)\right\|_{L_x^2}^2+\lambda^2\int_0^t \|E_{R,xt}\|_{L_x^2}^2\,dt \\
&\quad+\int_0^t\int_0^1\left(\left(n^0+p^0\right)+\left(n_B^0+p_B^0\right)+\lambda\left(n_B^1+p_B^1\right)+\lambda^2\left(n_B^2+p_B^2\right)\right)|E_{R,t}|^2\,dx\,dt \\
&=\frac{\lambda^2}{2}\left\|E_{R,t}(x,0)\right\|_{L_x^2}^2+\int_0^t\int_0^1(g_{1,t}+g_{2,t})E_{R,t}\,dx\,dt \\
&\quad-\int_0^t\int_0^1\left(\left(n^0+p^0\right)+\left(n_B^0+p_B^0\right)+\lambda\left(n_B^1+p_B^1\right)+\lambda^2\left(n_B^2+p_B^2\right)\right)_t E_R E_{R,t}\,dx\,dt. \qquad (4.29)
\end{aligned}$$

The second term in the right hand can be dealt with as

$$\begin{aligned}
\int_0^t\int_0^1(g_{1,t}+g_{2,t})E_{R,t}\,dx\,dt \leqslant{}& \epsilon\int_0^t \|E_{R,t}\|_{L_x^2}^2\,dt+M\int_0^t\left\|(z_R,z_{R,t})\right\|_{L_x^2}^2\,dt \\
&+M\left\|\xi\phi_{B,\xi t}^0\right\|_{L_{xt}^\infty}^2\int_0^t \|z_{R,x}\|_{L_x^2}^2\,dt+M\left\|\xi\phi_{B,\xi}^0\right\|_{L_{xt}^\infty}^2\int_0^t \|z_{R,xt}\|_{L_x^2}^2\,dt \\
&+M\int_0^t\left\|(z_R,z_{R,x})\right\|_{L_x^2}^2\|E_{R,t}\|_{L_x^2}^2+\left\|(z_{R,t},z_{R,xt})\right\|_{L_x^2}^2\|E_R\|_{L_x^2}^2\,dt \\
&+M\lambda. \qquad (4.30)
\end{aligned}$$

Here we have used the estimate

$$\begin{aligned}
&\int_0^t\int_0^1\frac{h}{\lambda}\left(\phi_{B,\xi}^0 z_R\right)_t E_{R,t}\,dx\,dt \\
&=\int_0^t\int_0^1 h\left(\frac{x}{\lambda}\phi_{B,\xi t}^0\frac{z_R}{x}+\frac{x}{\lambda}\phi_{B,\xi}^0\frac{z_{R,t}}{x}\right)E_{R,t}\,dx\,dt \\
&\leqslant\epsilon\int_0^t \|E_{R,t}\|_{L_x^2}^2\,dt+C(\epsilon)\left(\left\|\xi\phi_{B,\xi t}^0\right\|_{L_{xt}^\infty}^2\int_0^t \|z_{R,x}\|_{L_x^2}^2\,dt+\left\|\xi\phi_{B,\xi}^0\right\|_{L_{xt}^\infty}^2\int_0^t \|z_{R,xt}\|_{L_x^2}^2\,dt\right)
\end{aligned}$$

with the aid of Hardy–Littlewood's inequality.

For the third term (called I_3), one obtains that

$$I_3 \leqslant \epsilon \int_0^t \|E_{R,t}\|_{L_x^2}^2 \, dt + M \int_0^t \|E_R\|_{L_x^2}^2 \, dt. \tag{4.31}$$

Combining (4.29)–(4.31) and taking ϵ small enough, the above shows that

$$\begin{aligned}
&\lambda^2 \big\| E_{R,t}(t) \big\|_{L_x^2}^2 + \lambda^2 \int_0^t \|E_{R,xt}\|_{L_x^2}^2 \, dt + c_4 \int_0^t \|E_{R,t}\|_{L_x^2}^2 \, dt \\
&\quad \leqslant \lambda^2 \big\| E_{R,t}(x,0) \big\|_{L_x^2}^2 + M \int_0^t \big\| (z_R, z_{R,t}, E_R) \big\|_{L_x^2}^2 \, dt \\
&\qquad + M \big\| \xi \phi_{B,\xi t}^0 \big\|_{L_{xt}^\infty}^2 \int_0^t \|z_{R,x}\|_{L_x^2}^2 \, dt + M \big\| \xi \phi_{B,\xi}^0 \big\|_{L_{xt}^\infty}^2 \int_0^t \|z_{R,xt}\|_{L_x^2}^2 \, dt \\
&\qquad + M \int_0^t \Big(\big\| (z_R, z_{R,x}) \big\|_{L_x^2}^2 \|E_{R,t}\|_{L_x^2}^2 + \big\| (z_{R,t}, z_{R,xt}) \big\|_{L_x^2}^2 \|E_R\|_{L_x^2}^2 \Big) \, dt \\
&\qquad + M\lambda.
\end{aligned} \tag{4.32}$$

Then $\delta(4.28) + (4.32)$, by taking $\delta > 0$ small enough and $M\|\xi\phi_{B,\xi}^0\|_{L_{xt}^\infty}^2 \leqslant \frac{\delta c_3}{2}$, $\|\xi\phi_{B,\xi t}^0\|_{L_{xt}^\infty}^2 \leqslant 1$, $\|\phi_{B,\xi t}^0\|_{L_{xt}^\infty}^2 \leqslant 1$, we can get that $M\|\xi\phi_{B,\xi}^0\|_{L_{xt}^\infty}^2 \int_0^t \|z_{R,xt}\|_{L_x^2}^2 \, dt$ in the right-hand side can be absorbed by $\delta c_3 \int_0^t \|z_{R,xt}\|_{L_x^2}^2 \, dt$ in the left-hand side. Then we can obtain (4.22). The proof of Lemma 4 is completed. □

Finally, we estimate of the space derivatives $(z_{R,x}, E_{R,x})$.

Lemma 5. *Under the assumptions of Theorem* 1, *we have*

$$\begin{aligned}
\big\| (z_{R,x}, E_R) \big\|_{L_x^2}^2 + \lambda^2 \|E_{R,x}\|_{L_x^2}^2 &\leqslant M \big\| (z_R, z_{R,t}) \big\|_{L_x^2}^2 + M\lambda^2 \big\| (E_R, E_{R,t}) \big\|_{L_x^2}^2 + M\lambda^4 \|E_{R,x}\|_{L_x^2}^2 \\
&\quad + M\lambda^4 \big\| (E_R, E_{R,x}) \big\|_{L_x^2}^2 \|E_{R,x}\|_{L_x^2}^2 + M \big\| (z_R, z_{R,x}) \big\|_{L_x^2}^2 \|E_R\|_{L_x^2}^2 \\
&\quad + M\lambda.
\end{aligned} \tag{4.33}$$

Lemma 5 can be easily obtained from (4.16) and (4.20). We omit the proof here.

Proof of Theorem 1. We introduce the following two λ-weighted Liapunov-type functionals:

$$\Gamma^\lambda(t) = \big\| (z_R, z_{R,x}, z_{R,t}, E_R) \big\|_{L_x^2}^2 + \lambda^2 \big\| (E_R, E_{R,x}, E_{R,t}) \big\|_{L_x^2}^2 \tag{4.34}$$

and

$$G^\lambda(t) = \big\| (z_{R,x}, z_{R,xt}, E_R, E_{R,t}) \big\|_{L_x^2}^2 + \lambda^2 \big\| (E_{R,x}, E_{R,xt}) \big\|_{L_x^2}^2. \tag{4.35}$$

Then it follows from (4.10), (4.22) and (4.33), we have

$$\begin{aligned}\Gamma^\lambda(t) + \int_0^t G^\lambda(s)\,ds \leqslant M\Gamma^\lambda(t=0) + M\int_0^t \big(\Gamma^\lambda(s) + \big(\Gamma^\lambda(s)\big)^2\big)\,ds \\ + M\int_0^t \Gamma^\lambda(s)G^\lambda(s)\,ds + M\big(\Gamma^\lambda(t)\big)^2 + M\lambda, \qquad (4.36)\end{aligned}$$

thus, by Lemma 2, we can get

$$\Gamma^\lambda(t) \leqslant \tilde{M}\lambda^{\min\{\alpha,1\}-\delta} \qquad (4.37)$$

holds for any $\delta \in (0, \min\{\alpha, 1\})$ and $0 \leqslant t \leqslant T$, if $\Gamma^\lambda(0) \leqslant M\lambda^{\min\{\alpha,1\}}$.

Next we will prove that there exists a positive constant $\tilde{M}$ and $\alpha > 0$ such that

$$\Gamma^\lambda(t=0) \leqslant \tilde{M}\lambda^\alpha. \qquad (4.38)$$

In fact, from (4.2)–(4.5), (2.36), (2.37), we have

$$\big\|z_{R,t}(x,0)\big\|_{L^2_x} \leqslant M\lambda^{\frac{1}{2}}, \qquad \big\|\lambda E_{R,t}(x,0)\big\|_{L^2_x} \leqslant M\lambda^{\frac{1}{2}}, \qquad (4.39)$$

then

$$\begin{aligned}\Gamma^\lambda(t=0) &= \big\|z_{R,t}(x,0)\big\|^2_{L^2_x} + \big\|\lambda E_{R,t}(x,0)\big\|^2_{L^2_x} \\ &\leqslant \tilde{M}\lambda. \qquad (4.40)\end{aligned}$$

With $\alpha = 1$, one gets (2.39). This completes the proof of Theorem 1. □

Proof of Theorem 2. The proof of Theorem 2 is obvious. □

Proof of Theorem 3. To perform the energy estimates, we derive some boundary conditions for the error functions z_R, E_R.

Since $D'(x = 0, 1) = 0$, one gets from (2.25) that

$$\varepsilon(x = 0, 1) = 0$$

and hence from (2.27),

$$z_x(x = 1) = 0,$$

thus we have

$$z_R(x = 0) = 0, \qquad z_{R,x}(x = 1) = 0.$$

Then we can make the energy estimates in the similar way as above and get the result of Theorem 2 finally. We omit this. □

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References

[1] Y. Brenier, Convergence of the Vlasov–Poisson system to the incompressible Euler equations, Comm. Partial Differential Equations 25 (2000) 737–754.
[2] L. Caffarelli, J. Dolbeault, P.A. Markowich, C. Schmeiser, On Maxwellian equilibria of insulted semiconductors, Interfaces Free Bound. 2 (2000) 331–339.
[3] J. Dolbeault, P.A. Markowich, A. Unterreiter, On singular limits of mean-field equations, Arch. Ration. Mech. Anal. 158 (2001) 319–351.
[4] H. Gajewski, On existence, uniqueness and asymptotic behavior of solutions of the basic equations for carrier transport in semiconductors, ZAMM Z. Angew. Math. Mech. 65 (2) (1985) 101–108.
[5] I. Gasser, L. Hsiao, P. Markowich, et al., Quasineutral limit of a nonlinear drift-diffusion model for semiconductor models, J. Math. Anal. Appl. 168 (2002) 184–199.
[6] L. Hsiao, Q.C. Ju, S. Wang, Quasi-neutral limit of the drift-diffusion models for semiconductors: the case of general sign-changing doping profile, Sci. China 38 (3) (2008) 286–296.
[7] L. Hsiao, S. Wang, Quasineutral limit of a time-dependent drift-diffusion-Poisson model for p-n junction semiconductor devices, J. Differential Equations 225 (2) (2006) 411–439.
[8] A. Jüngel, Y.J. Peng, A hierarchy of hydrodynamic models for plasma: quasineutral limits in the drift-diffusion equations, Asymptot. Anal. 28 (2001) 49–73.
[9] Q.C. Ju, S. Wang, Quasi-neutral limit of the multidimensional drift-diffusion models for semiconductors, Math. Models Methods Appl. Sci. (2010), in press.
[10] P.A. Markowich, A singular perturbation analysis of the fundamental semiconductor device equations, SIAM J. Appl. Math. 44 (1984) 896–928.
[11] P.A. Markowich, C.A. Ringhofer, A singularly perturbed boundary value problem modelling a semiconductor device, SIAM J. Appl. Math. 44 (1984) 231–256.
[12] P.A. Markowich, C. Ringhofer, C. Schmeiser, Semiconductors Equations, Springer-Verlag, New York, 1990.
[13] P.A. Markowich, C.A. Ringhofer, C. Schmeiser, An asymptotic analysis of one-dimensional models of semiconductor devices, IMA J. Appl. Math. 37 (1986) 1–24.
[14] C. Schmeiser, S. Wang, Quasineutral limit of the drift-diffusion model for semiconductors with general initial data, Math. Models Methods Appl. Sci. 13 (2003) 463–470.
[15] W. Shockley, The theory of p-n junctions in semiconductors and p-n junction transistors, Bell. Syst. Tech. J. 28 (1949) 435–489, reprinted in Ann. Mat. Pura Appl. 166 (1987) 65–96.
[16] R.A. Smith, Semiconductors, second ed., Cambridge University Press, Cambridge, UK, 1978.
[17] S.M. Sze, Physics of Semiconductor Devices, Wiley–Interscience, New York, 1969.
[18] S. Wang, Quasineutral limit of Euler–Poisson system with and without viscosity, Comm. Partial Differential Equations 29 (2004) 419–456.
[19] S. Wang, Quasineutral limit of the multi-dimensional drift-diffusion-Poisson models for semiconductors with PN-junctions, Math. Models Methods Appl. Sci. 16 (4) (2006) 537–557.
[20] S. Wang, Z.P. Xin, P.A. Markowich, Quasi-neutral limit of the drift-diffusion models for semiconductors: the case of general sign-changing doping profile. (I), SIAM J. Math. Anal. 37 (6) (2006) 1854–1889.